Higher Cats RG #14 (11/19)

Refs: [Cisinski] 4.2, 4.3
[HTT]
[Rezk: Intro to Quasicats].
Stuff on Quasicats.

§4.2: Mapping spaces as fibers + §4.3 Final objects.

Recall the join of ssets X, Y

$X * Y = X \star Y$ — $(X \star Y)_n = \coprod_{i+j+1=n} X_i \times Y_j$

ie. an n-simplex $\Delta^n \to X \star Y$ consists of
- an i-simplex in X
- a j-simplex in Y

This forms ⊛ functors for any sset X ...
$- \star X$, $X \star -$: $sSet \rightrightarrows sSet_{X/}$

$$\begin{tikzcd} X \arrow[d, hook] \arrow[r] & Y \star X . \\ X \star Y & \end{tikzcd}$$

These have R-adjts called slices

$$\begin{tikzcd} sSet \arrow[r, "- \star X"] & sSet_{X/} \arrow[l, bend left, "slice"] \end{tikzcd} \qquad \begin{tikzcd} sSet \arrow[r, "X \star -"] & sSet_{X/} \arrow[l, bend left, "slice"] \end{tikzcd}$$

$$\begin{tikzcd} S_{/p} & X \arrow[d, "p"] \arrow[l, mapsto] \\ & S \end{tikzcd} \qquad \begin{tikzcd} S_{p/} & X \arrow[d, "p"] \arrow[l, mapsto] \\ & S \end{tikzcd}$$

$S_{/p,n} = \underset{sSet}{\mathrm{Hom}}(\Delta^n, S_{/p}) \cong \underset{sSet_{X/}}{\mathrm{Hom}}(\Delta^n \star X, S)$

$$\begin{tikzcd} X \arrow[d, "i"] \arrow[dr, "p"] & \\ \Delta^n \star X & S \end{tikzcd}$$

ie. maps

$$\begin{tikzcd} \Delta^n \star X \arrow[r] & S \\ X \arrow[u, hook] \arrow[ur, bend right, ""] \arrow[ur, phantom, "="] & \end{tikzcd}$$

Note: any $S_{/p}$ comes w/ a projection $S_{/p} \to S$

$$\begin{tikzcd} \Delta^n \star X \arrow[r] & S \\ \Delta^n \arrow[u, hook] \arrow[ur, bend right] & \end{tikzcd}$$

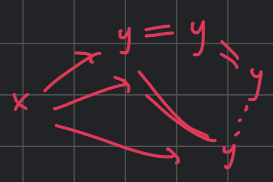
This lets us describe a "mapping space" in an ∞-cat. as a pullback (fiber)

Def. Let $\mathcal{C} \in qCat$.
$x, y \in \mathcal{C}_0$
Define a sset

$$\begin{tikzcd} \mathrm{Map}^L_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_{/y} \arrow[d] \\ \Delta^0 \arrow[r, "x"'] & \mathcal{C} \end{tikzcd} \quad \cdots \quad \begin{tikzcd} \mathrm{Map}^R_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"'] & \mathcal{C} \end{tikzcd}$$

Prop [HTT, 1.2.2.3]: Both of these are Kan complexes.

Rk: Can think of $\mathrm{Map}^L_{\mathcal{C}}(x,y)$ as having n-simplices are of the form



Rk: Can show that these are "weakly equiv." to eachother & both equivalent to the mapping space we described earlier

$$\begin{tikzcd} \mathcal{C}(x,y) = \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d, "{(s,t)}"] \\ \Delta^0 \arrow[r, "{(x,y)}"'] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

[HTT, 4.2.1.8]

[C, 4.2.10]

$$\mathrm{Map}^L_e(x,y) \xrightarrow{\sim} \mathrm{Map}_e(x,y) \xleftarrow{\sim} \mathrm{Map}^R_e(x,y)$$

To do this, it's useful to describe alternate slice/join constructions, which are in some sense equiv. to the ordinary ones. (see [Rezk, §10]).

Def : (alt. join). $X, Y \in s\mathrm{Set}$, the alternate join $X \diamond Y \in s\mathrm{Set}$

$$\begin{tikzcd} X \times \partial\Delta^n \times Y \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X \sqcup Y \arrow[d] \\ X \times \Delta^1 \times Y \arrow[r] & X \diamond Y \end{tikzcd}$$

Def. (alt. slice) Similarly as before, can show this is functorial in both variables, and have R-adjoints which we'll call alt. slices

$$\begin{tikzcd} s\mathrm{Set} \arrow[r, shift left, "- \diamond X"] & s\mathrm{Set}_{X/} \arrow[l, shift left, "\text{alt. slice}"] \end{tikzcd} \qquad \begin{tikzcd} s\mathrm{Set} \arrow[r, shift left, "X \diamond -"] & s\mathrm{Set}_{X/} \arrow[l, shift left, "\text{alt slice}"] \end{tikzcd}$$

$$S/\!\!/p = S^{/p} \leftarrow\!\shortmid \begin{matrix} X \\ \downarrow p \\ S \end{matrix} \qquad p/\!\!/S = S^{p/} \leftarrow\!\shortmid \begin{matrix} X \\ \downarrow p \\ S \end{matrix}$$

Prop : There's a weak categorical equiv. $X \diamond Y \xrightarrow{\tau_{X,Y}} X \star Y$

w.e. in $s\mathrm{Set}_{\mathrm{Joyal}}$

Prop : For $X \in q\mathrm{Cat}$, $T \xrightarrow{t} X$ a map of ssets, there's an equiv. of ∞-cats

$$X_{/t} \xrightarrow{\sim} X^{/t}$$

↳ Lets you describe a mapping space as a fiber of $X_{/y}$ —
& a mapping space " " over $X^{/y}$

↳ compare them ⇝ show that both are equiv. to the usual mapping space.

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§4.3: Final objects.

[C, 4.1.5]

Recall in §4.1, we described a model structure on $s\mathrm{Set}_{/S}$ for a $S \in s\mathrm{Set}$, we called the contravariant model structure : $s\mathrm{Set}_{/S,\mathrm{contra}}$. — w.e: $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$ & $\begin{matrix} Y \\ \downarrow \\ S \end{matrix}$ are weak equiv. if

$X^{\triangleright} = X \star \Delta^0$

the induced map

$$X^{\triangleright} \cup_X S \longrightarrow Y^{\triangleright} \cup_Y S$$

is a weak cat'l equiv.

[HTT, 2.1.4].

cof = monos

fib = R-fibs.

Recall: A map $X \to S$ of ssets is called final if $\forall$ sset T & mor. $X \to T$

$$\begin{pmatrix} S \\ \downarrow \\ T \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ \downarrow \\ T \end{pmatrix}$$ is a weak equiv. in $s\mathrm{Set}_{/T,\mathrm{contra}}$.

Def. Let $X \in sSet$. An object $x \in X_0$ is a final object if the map $\Delta^0 \xrightarrow{x} X$ is final in the sense above.

i.e. $\forall$ map of ssets $X \xrightarrow{\varphi} S$, the map

$$\begin{pmatrix} \Delta^0 \\ \downarrow \varphi\circ x \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ \downarrow \varphi \\ S \end{pmatrix}$$ is a weak eq. in $sSet_{/S}$, contra.

Rk. ($x \in X_0$ is final) iff ($\Delta^0 \xrightarrow{x} X$ is a R-anodyne ext.)

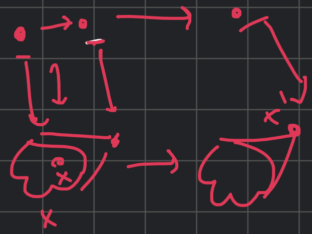
Prop: Let $f : X \longrightarrow Y$ be a map of ssets
$x \in X_0$ a final object in X.

Then (f is final as a map ssets) iff ($f(x)$ is a final object in Y)

Pf: Follows from closure properties (see [C, 4.1.9 (a), (b)]) ▨

Def. Let (X, x) be a ptd sset.
Define a ptd sset $C(X) = (\underset{\overset{\shortparallel}{X\star\Delta^0}}{X^{\triangleright}}, x')$ where x' is ...

$$\begin{tikzcd} \Delta^1 \cong \Delta^0 \star \Delta^0 \arrow[r] \arrow[d, "x \star id"'] \arrow[dr, phantom, "\ulcorner", very near end] & \Delta^0 \arrow[d, "x'"] \\ X^{\triangleright} \cong X \star \Delta^0 \arrow[r] & C(X) \end{tikzcd}$$



This forms a functor $C : sSet_* \longrightarrow sSet_*$

Prop: The object x' is a final object in $C(X)$.

Pf (skipped).

The functor c has a R-adjt $\mathrm{sSet}_* \longrightarrow \mathrm{sSet}_*$
$(Y,y) \longmapsto (Y_{/y}, \mathrm{id}_y)$

ie. $\underset{\mathrm{sSet}_*}{\mathrm{Hom}}(c(X)_x, Y_y) \cong \underset{\mathrm{sSet}_*}{\mathrm{Hom}}(X_x, {Y_{/y}}_{\mathrm{id}_y})$

Prop: Let $X \in \mathrm{sSet}$, $x \in X_0$ an object

If there's a section s

$$\begin{tikzcd} X_{/x} \arrow[d] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

s.t. $s(x) = \mathrm{id}_x$

$\Rightarrow$ x is a final object.

If $X \in \mathrm{qCat}$, then this is an iff.

Pf: (skipped)

Cor (4.3.8): Let $X \in \mathrm{sSet}$, $x \in X_0$.

Then id_x is final in $X_{/x}$.

Fibrations are R-fibrations.

(one version of Yoneda)

↳ Thm: Let $X \in \mathrm{qCat}$. $x \in X_0$. The object $\begin{pmatrix}\Delta^0\\ \downarrow x\\ X\end{pmatrix}$ in $\mathrm{sSet}_{/X,\mathrm{contra}}$ has fibrant replacement $\begin{pmatrix}X_{/x}\\ \downarrow e_x\\ X\end{pmatrix}$.

In particular, $\forall\, y \in X_0$, there's an equiv. of ∞-gpds

$$\mathrm{map}_X(X_{/x}, X_{/y}) = \mathrm{Map}_{/X}(X_{/x}, X_{/y}) \xrightarrow{\sim} \mathrm{Map}_X(x,y)$$

$\overset{\shortparallel}{X(x,y)}$

$$\begin{tikzcd} \mathrm{Map}_{/X}(X_{/x}, X_{/y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}(X_{/x}, X_{/y}) \arrow[d, "e_{y*}"] \\ \Delta^0 \arrow[r, "e_x"'] & \mathrm{Hom}(X_{/x}, X) \end{tikzcd}$$

[4.1.12]

This is one form of Yoneda lemma

Pf. We saw that id_x is a final object in $X_{/x}$

$$\begin{tikzcd} \Delta^0 \arrow[rr, "\mathrm{id}_x"] \arrow[dr, "x"'] & & X_{/x} \arrow[dl, "e"] \\ & X & \end{tikzcd}$$

id_x is final $\Longleftrightarrow$ there's a w.eq. $\begin{pmatrix}\Delta^0\\ \downarrow x\\ X\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}X_{/x}\\ \downarrow\\ X\end{pmatrix}$ in $\mathrm{sSet}_{/X,\mathrm{contra}}$.

$\Rightarrow \forall$ R-fibrant $\begin{pmatrix}X_{/y}\\ e_y\downarrow\\ X\end{pmatrix}$, the induced map

[4.1.14] $\mathrm{Map}_{/X}(\Delta^0, X_{/y}) \xrightarrow{\sim} \mathrm{Map}_{/X}(X_{/x}, X_{/y})$

is an eq. of ∞-gpds.

$$\begin{tikzcd}
& \mathrm{Map}_{/x}(\Delta^0, X_{/y}) \arrow[dl] \arrow[dr] \arrow[rr, "\sim"] \arrow[dd, phantom, "\checkmark"] & & \mathrm{Map}_{/x}(X_{/x}, X_{/y}) \arrow[r, dash, bend left, "\sim"] & \mathrm{Map}_X(x,y) \\
\Delta^0 \arrow[dr, "x"'] & & \underline{\mathrm{Hom}}(\Delta^0, X_{/y}) \arrow[dl, "p_{y*}"] \arrow[r, phantom, "\simeq"] & X_{/y} \arrow[dl, "p_y"] & \\
& \underline{\mathrm{Hom}}(\Delta^0, X) \arrow[r, phantom, "\simeq"] & X & &
\end{tikzcd}$$

$$\Rightarrow \mathrm{Map}_{/x}(\Delta^0, X_{/y}) \simeq X_{/y} \times_x \Delta^0 \underset{\boxed{4.2.10}}{\simeq} \mathrm{Map}_X(x,y)$$

☒

Thm (Joyal) Let $X \in \mathrm{qCat}$
$x \in X_0$
Denote the map $X_{/x} \xrightarrow{\pi} X$

TFAE:
(1) x is a final object in X
(2) $\forall\, y \in X_0$, the mapping space $\mathrm{Map}_X(x,y)$ is contractible.

(3) π is a trivial fibration
(4) " " equiv. of ∞-cats.
(5) π has a section sending $x \mapsto \mathrm{id}_x$
(6) Any bdy $\partial\Delta^n \to X$ fills to a simplex s.t.
$(n>0)$

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "u"] \arrow[d, hook] & X \\
\Delta^n \arrow[ur, bend right, "\bar{u}"'] &
\end{tikzcd}$$

$\bar{u}(n) = x.$

Pf. (will include)

Cor: $x \in X$ final $\Rightarrow$ $x \in hX$ final

$\boxed{\infty\text{-cat.}}$

$\not\Leftarrow$

not true.

Cor: Final objects in an ∞-cat. form an ∞-gpd which is either $\emptyset$ or contractible.

final objects are unique.

⋮

some other stuff.

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